ABSTRACT. For a Banach space $X$ and an increasing subadditive continuous function $\varphi$ on $[0, \infty)$ with $\varphi(0) = 0$, let us denote by $L^\varphi(I, X)$, the space of all $X$-valued $\varphi$-integrable functions $f : I \to X$ on a certain positive complete $\sigma$-finite measure space $(I, \sum, \mu)$ with $\int_I \varphi \|f(t)\| \, d\mu(t) < \infty$ and $l^\varphi(X) = \left\{ (x_k) : \sum_{k=1}^\infty \varphi \|x_k\| < \infty, \ x_k \in X \right\}$.

The aim of this paper is to prove that for a closed separable subspace $G$ of $X$, $L^\varphi(I, G)$ is simultaneously proximinal in $L^\varphi(I, X)$ if and only if $G$ is simultaneously proximinal in $X$. Other result on simultaneous approximation of $l^\varphi(G)$ in $l^\varphi(X)$ is presented.

1. INTRODUCTION

A function $\varphi : [0, \infty) \to [0, \infty)$ is called a modulus function if it satisfies the following conditions:

(1) $\varphi$ is continuous and increasing function.
(2) $\varphi(x) = 0$ if and only if $x = 0$.
(3) $\varphi(x + y) \leq \varphi(x) + \varphi(y)$.

The functions $\varphi(x) = x^p$, $0 < p < 1$, and $\varphi(x) = \ln(1 + x)$ are modulus functions. In fact if $\varphi$ is a modulus function, then $\psi(x) = \varphi(x)/(1 + \varphi(x))$ is a modulus function. Further the composition of two modulus function is a modulus function.

For a modulus function $\varphi$ and a Banach space $X$, let us denote by $L^\varphi(I, X)$, the space of all $X$-valued $\varphi$-integrable functions $f : I \to X$ on a certain positive complete $\sigma$-finite measure space $(I, \sum, \mu)$ with $\int_I \varphi \|f(t)\| \, d\mu(t) < \infty$ and

$$l^\varphi(X) = \left\{ (x_k) : \sum_{k=1}^\infty \varphi \|x_k\| < \infty, \ x_k \in X \right\}.$$  

For $a = (a_k) \in l^\varphi(X)$ and $f \in L^\varphi(I, X)$ set

$$\|a\|_\varphi = \sum_{k=1}^\infty \varphi \|a_k\| \quad \text{and} \quad \|f\|_\varphi = \int_I \varphi \|f(t)\| \, d\mu(t).$$

If $X = C$, the set of complex numbers, the spaces $l^\varphi(X)$ and $L^\varphi(I, X)$ is simply denoted by $l^\varphi$ and $L^\varphi(I)$ respectively. It is known, [4], that $l^\varphi \subseteq l^1$, $L^\varphi(I) \supseteq L^1(I)$.
and \((l^p(X), \|\cdot\|_p)\) and \((L^p(I, X), \|\cdot\|_p)\) are complete metric linear spaces. For more on \(l^p\) and \(L^p(I)\) we refer to the reader to [3] and [5].

Note that the Banach space \(X\) is a metric space with the metric \(d(x, y) = \varphi \|x - y\|\).

**Definition 1.1.** Let \(\varphi\) be a modulus function and \(G\) be a closed subspace of a Banach space \(X\). We say that

(a) \(G\) is simultaneously proximinal in \(X\) if for each \(m\)-tuple of elements \((x_1, x_2, \ldots, x_m) \in X^m\) there exists \(g \in G\) such that:

\[
\sum_{i=1}^{m} \varphi \|x_i - g\| = \text{dist}_\varphi(x_1, x_2, \ldots, x_m, G) = \inf_{h \in G} \sum_{i=1}^{m} \varphi \|x_i - h\|.
\]

In other words for every \(h \in G\)

\[
\|\sum_{i=1}^{m} \varphi (x_i - g)\| \leq \|\sum_{i=1}^{m} \varphi (x_i - h)\|.
\]

(b) \(L^p(I, G)\) is simultaneously proximinal in \(L^p(I, X)\) if for each \(m\)-tuple of elements \(f_1, f_2, \ldots, f_m \in (L^p(I, X))^m\) there exists \(g \in L^p(I, G)\) such that

\[
\sum_{i=1}^{m} \|f_i - g\| = \text{dist}_\varphi(f_1, f_2, \ldots, f_m, L^p(I, G)) = \inf_{h \in L^p(I, G)} \sum_{i=1}^{m} \|f_i - h\|.
\]

The problem of best simultaneous approximation has been studied by many authors e.g., [2], [9], [14] and [15]. Most of these works have dealt with the characterization of best simultaneous approximation in spaces of continuous functions with values in a Banach space \(X\). Some existence and uniqueness results were obtained. Results on best simultaneous approximation in general Banach spaces may be found in [11] and [13].

Related results on \(L^p(I, X), 1 \leq p < \infty\), are given in [14]. In [14], it is shown that if \(G\) is a reflexive subspace of a Banach space \(X\), then \(L^p(I, G)\) is simultaneously proximinal in \(L^p(I, X)\). If \(p = 1\), Abu Sarhan and Khalil [1], proved that if \(G\) is a reflexive subspace of the Banach space \(X\) or \(G\) is a 1-summand subspace of \(X\), then \(L^1(I, G)\) is simultaneously proximinal in \(L^1(I, X)\).

The aim of this paper is to prove that for a closed separable subspace \(G\) of \(X\), \(L^1(I, G)\) is simultaneously proximinal in \(L^1(I, X)\) if and only if \(G\) is simultaneously proximinal in \(X\). Some results are inspired by the results in [14]. Other result on simultaneous approximation of \(l^p(G)\) in \(l^p(X)\) is presented.

Throughout this paper, \((I, \sum, \mu, \cdot\cdot\cdot)\) is a \(\sigma\)-finite measure space, \(X\) is a Banach space, \(G\) is a closed subspace of \(X\) and the norm of \(v \in X\) is denoted by \(\|v\|\).

### 2. Distance Formulae

Progress in the discussion of simultaneous proximality when \(X\) does not possess pleasant properties is greatly facilitated by the fact that the distance from an \(m\)-tuple of elements \(f_1, f_2, \ldots, f_m \in L^p(I, X)\) to a subspace \(L^p(I, G)\) is computed by the following theorem:
Theorem 2.1. Let \( \varphi \) be a modulus function and \( f_1, f_2, ..., f_m \in L^\varphi (I, X) \). Then

\[
\text{dist}_\varphi (f_1, f_2, ..., f_m, L^\varphi (I, G)) = \int_I \text{dist}_\varphi (f_1(s), f_2(s), ..., f_m(s), G) \, d\mu(s).
\]

Proof. Let \( f_1, f_2, ..., f_m \in L^\varphi (I, X) \). Then for each \( i = 1, 2, ..., m \), \( f_i \) is the limit almost everywhere of a sequence of simple functions \( \{f_{i,n}\} \) in \( L^\varphi (I, X) \). Since the distance function \( \text{dist}_\varphi (x, G) \) is continuous in \( x \in X \), \( \lim_{n \to \infty} \varphi (\|f_{i,n}(s) - f_i(s)\|) = 0 \), \( i = 1, 2, ..., m \), implies that

\[
\lim_{n \to \infty} \left| \text{dist}_\varphi (f_{1,n}(s), ..., f_{m,n}(s), G) - \text{dist}_\varphi (f_1(s), ..., f_m(s), G) \right| = 0.
\]

Furthermore for each \( n \), the function: \( s \mapsto \text{dist}_\varphi (f_{1,n}(s), f_{2,n}(s), ..., f_{m,n}(s), G) \) is a simple function and so we may assume that \( \text{dist}_\varphi (f_1(s), f_2(s), ..., f_m(s), G) \) is measurable. Now for any \( g \in L^\varphi (\mu, G) \)

\[
\int_I \text{dist}_\varphi (f_1(s), f_2(s), ..., f_m(s), G) \, d\mu(s) \leq \int_I \sum_{i=1}^m \varphi (\|f_i(s) - g(s)\|) \, d\mu(s)
\]

\[
= \sum_{i=1}^m \int_I \varphi (\|f_i(s) - g(s)\|) \, d\mu(s).
\]

Therefore

\[
(1) \quad \int_I \text{dist}_\varphi (f_1(s), f_2(s), ..., f_m(s), G) \, d\mu(s) \leq \text{dist}_\varphi (f_1, f_2, ..., f_m, L^\varphi (I, G)).
\]

For the reverse inequality fix \( \epsilon > 0 \). Since simple functions are dense in \( L^\varphi (I, X) \), there exist simple functions, \( f_j \) in \( L^\varphi (I, X) \) such that \( \|f_j - f_j'\|_\varphi < \frac{\epsilon}{mn} \), \( j = 1, 2, ..., m \). Assume that \( f_j(t) = \sum_{i=1}^n \chi_{A_i}(t)y^{j}_i \), \( j = 1, 2, ..., m \), where \( \chi_{A_i} \) are the characteristic functions of the measurable sets \( A_i \) in \( I \) and \( y^{j}_i \in X \). We can assume that \( \sum_{i=1}^n \chi_{A_i} = 1 \) and \( \mu(A_i) > 0 \).

Given \( \epsilon > 0 \) for each \( i = 1, 2, ..., n \), select \( g_i \in G \) such that:

\[
\sum_{j=1}^m \varphi \|y^{j}_i - g_i\| < \text{dist}_\varphi (y^{1}_i, y^{2}_i, ..., y^{m}_i, G) + \frac{\epsilon}{n\mu(A_i)}.
\]
Let \( g(t) = \sum_{i=1}^{n} \chi_{A_i}(t)g_i \). Clearly \( g \in L^\varphi(I,G) \) and
\[
dist_{\varphi} \left( f_1, \ldots, f_m, L^\varphi(I,G) \right) \leq \sum_{j=1}^{m} \left\| f_j - f'_j \right\|_{\varphi} \\
+ \dist_{\varphi} \left( f'_1, f'_2, \ldots, f'_m, L^\varphi(I,G) \right) \\
\leq \epsilon + \sum_{j=1}^{m} \left\| f'_j - g \right\|_{\varphi} \\
= \epsilon + \sum_{j=1}^{m} \int_{I} \varphi \left\| f'_j(s) - g(s) \right\| d\mu(s) \\
= \epsilon + \sum_{j=1}^{m} \sum_{i=1}^{n} \int_{A_i} \varphi \left\| f'_j(s) - g(s) \right\| d\mu(s) \\
= \epsilon + \sum_{j=1}^{m} \sum_{i=1}^{n} \left( \varphi \left\| y'_j - g_i \right\| \right) \mu(A_i) \\
= \epsilon + \sum_{i=1}^{n} \sum_{j=1}^{m} \left( \varphi \left\| y'_j - g_i \right\| \right) \mu(A_i) \\
\leq \epsilon + \sum_{i=1}^{n} \mu(A_i) \dist_{\varphi} \left( y^1_i, y^2_i, \ldots, y^m_i, G \right) + \frac{\epsilon}{n} \\
\leq 2\epsilon + \sum_{i=1}^{n} \int_{A_i} \dist_{\varphi} \left( y^1_i, y^2_i, \ldots, y^m_i, G \right) d\mu(s) \\
= 2\epsilon + \int_{I} \dist_{\varphi} \left( f'_1(s), f'_2(s), \ldots, f'_m(s), G \right) d\mu(s).
\]

Since
\[
dist_{\varphi} \left( f'_1(s), f'_2(s), \ldots, f'_m(s), G \right) \leq \dist_{\varphi} \left( f_1(s), f_2(s), \ldots, f_m(s), G \right) \\
+ \sum_{j=1}^{m} \varphi \left\| f'_j(s) - f(s) \right\|.
\]
then,
\[
\text{dist}_\varphi (f_1, f_2, ..., f_m, L^r(I, G)) \leq 2\epsilon + \sum_{j=1}^{m} \int_I \varphi \left\| f_j'(s) - f_j(s) \right\| d\mu(s) \\
+ \int_I \text{dist}_\varphi (f_1(s), f_2(s), ..., f_m(s), G) d\mu(s) \\
= 2\epsilon + \sum_{j=1}^{m} \left\| f_j - f_j' \right\|_\varphi \\
+ \int_I \text{dist}_\varphi (f_1(s), f_2(s), ..., f_m(s), G) d\mu(s) \\
\leq 3\epsilon + \int_I \text{dist}_\varphi (f_1(s), f_2(s), ..., f_m(s), G) d\mu(s),
\]
which (since \(\epsilon\) is arbitrary) implies that
\[
(2) \quad \text{dist}_\varphi (f_1, f_2, ..., f_m, L^r(I, G)) \leq \int_I \text{dist}_\varphi (f_1(s), f_2(s), ..., f_m(s), G) d\mu(s).
\]
Hence by 1 and 2 the proof is complete. \(\square\)

An application of Theorem 2.1 is

**Corollary 2.2.** An element \(g \in L^r(I, G)\) is a best simultaneous approximation of \(f_1, f_2, ..., f_m \in L^r(I, X)\) if and only if \(g(t)\) is a best simultaneous approximation of \(f_1(t), f_2(t), ..., f_m(t) \in X\) for almost all \(t \in I\).

### 3. Best Simultaneous Approximation in \(L^r(I, X)\)

The main result in this section is, for a modulus function \(\varphi\) and a closed separable subspace \(G\) of a Banach space \(X\), \(L^r(I, G)\) is simultaneously proximinal in \(L^r(I, X)\) if and only if \(G\) is simultaneously proximinal in \(X\). We begin with the following:

**Theorem 3.1.** If \(G\) is simultaneously proximinal in \(X\), then for every \(m\)-tuple of simple function \(f_1, f_2, ..., f_m \in L^r(I, X)\), \(P(f_1, f_2, ..., f_m, L^r(I, X))\) is not empty, where \(P(f_1, f_2, ..., f_m, L^r(I, X))\) is the set of all elements \(g \in L^r(I, G)\) such that \(g\) is a best simultaneous approximation of \(m\)-tuple of the elements \(f_1, f_2, ..., f_m\).

**Proof.** Let \(f_1, f_2, ..., f_m\) be an \(m\)-tuple of simple functions in \(L^r(I, X)\). With no loss of generality we can assume that \(f_j(t) = \sum_{i=1}^{n} \chi_{A_i}(t)y_{ij}\), where \(A_i\) are disjoint measurable sets such that \(\bigcup_{i=1}^{n} A_i = I\). Pick \(g_i \in G\) such that \(g_i\) is a best simultaneous approximation of
the m-tuple of elements \( y_i^1, y_i^2, \ldots, y_i^m \in X, i = 1, 2, \ldots, n \). Set \( g(t) = \sum_{i=1}^n \kappa A_i(t) g_i \). Then for any \( h \in L^\varphi(I, X) \) we have:

\[
\sum_{j=1}^m \| f_j - h \| \varphi = \sum_{j=1}^m \int_I \varphi \| f_j(s) - h(s) \| d\mu(s)
\]

\[
= \int_I \sum_{j=1}^m \varphi \| f_j(s) - h(s) \| d\mu(s)
\]

\[
= \sum_{i=1}^n \int_{A_i} \sum_{j=1}^m \varphi \| y_i^j - h_i \| d\mu(s)
\]

\[
\geq \sum_{i=1}^n \int_{A_i} \sum_{j=1}^m \varphi \| y_i^j - g_i \| d\mu(s)
\]

\[
= \int_I \sum_{j=1}^m \varphi \| f_j(s) - g(s) \| d\mu(s).
\]

Hence \( \sum_{j=1}^m \| f_j - g \| \varphi = \inf_{h \in L^\varphi(I, G)} \sum_{j=1}^m \| f_j - h \| \varphi \). □

**Theorem 3.2.** If \( \varphi \) is a modulus function, then \( G \) is simultaneously proximinal in \( X \) if \( L^\varphi(I, G) \) is simultaneously proximinal in \( L^\varphi(I, X) \).

**Proof.** Let \( x_1, x_2, \ldots, x_m \in X \). Set \( f_j = 1 \otimes x_j, j = 1, 2, \ldots, m \), where 1 is the constant function 1. Clearly for each \( j = 1, 2, \ldots, m \), \( f_j \in L^\varphi(I, X) \). By assumption there exists \( g \in L^\varphi(I, G) \) such that for any \( h \in L^\varphi(I, G) \)

\[
\sum_{j=1}^m \| f_j - g \| \varphi \leq \sum_{j=1}^m \| f_j - h \| \varphi.
\]

By Theorem 2.1

\[
\sum_{j=1}^m \varphi \| f_j(t) - g(t) \| \leq \sum_{j=1}^m \varphi \| f_j(t) - h(t) \|
\]

a.e. in \( I \). Or

\[
\sum_{j=1}^m \varphi \| x_j - g(t) \| \leq \sum_{j=1}^m \varphi \| x_j - h(t) \|.
\]

Let \( h \) run over all functions \( 1 \otimes z \), for \( z \in G \), we get

\[
\sum_{j=1}^m \varphi \| x_j - g(t) \| \leq \sum_{j=1}^m \varphi \| x_j - z \|.
\]

□
Now we pose the following problem: If $G$ is separable is it true that $L^*(I,G)$ is simultaneously proximinal in $L^*(I,X)$? to solve this problem we begin by the following:

**Lemma 3.3.** [Lemma 2.9 of [9]] Assume $\mu(I) < +\infty$. Suppose $(M,d)$ is a metric space and $A$ is a subset of $I$ such that $\mu^*(A) = \mu(I)$, where $\mu^*$ denotes the outer measure associated to $\mu$. If $g$ is a mapping from $I$ to $M$ with separable range, then for any $\epsilon > 0$ there exists a countable partition $\{E_n\}$ of $I$ in measurable sets and $A_n \subset A \cap E_n$ such that $\mu^*(A_n) = \mu(E_n)$ and $\text{diam}(g(A_n)) < \epsilon$ for all $n$.

**Theorem 3.4.** Let $G$ be a closed separable subspace of $X$. Let us suppose that $G$ is simultaneously proximinal in $X$ and $f_1, f_2, ..., f_m : I \to X$ be measurable functions. Then there is a measurable function $g : I \to X$ such that $g(t)$ is a best simultaneous approximation of $(f_1(t), f_2(t), ..., f_m(t))$ in $G$ for almost all $t$.

**Proof.** Let $f_1, f_2, ..., f_m : I \to X$ be measurable functions. So we may assume that $f_1(I), f_2(I), ..., f_m(I)$ are separable sets in $X$. Using the fact that $\mu$ is $\sigma$-finite we can find countable partitions $\{I_{1n}\}_{n=1}^\infty, \{I_{2n}\}_{n=1}^\infty, ..., \{I_{mn}\}_{n=1}^\infty$ of $I$ in measurable sets such that $\text{diam}_\sigma(f_i(I_{in})) < \frac{1}{2}$ and $\mu(I_{in}) < \infty$, $i = 1, 2, ..., m$, for all $n$, where

$$\text{diam}_\sigma A = \sup \{ \varphi \|x - y\| : x, y \in A \}.$$  

Consider the partition $\{I_{n_1,n_2,...,n_m}\}_{n_i=1}^\infty$, where $I_{n_1,n_2,...,n_m} = \bigcap_{i=1}^m I_{in}$, for $1 \leq n_i < \infty$. Then $\text{diam}_\sigma(f_i(I_{n_1,n_2,...,n_m})) < \frac{1}{2}$, $i = 1, 2, ..., m$. For simplicity we write $\{I_{n_1,n_2,...,n_m}\}_{n_i=1}^\infty$ as $\{I_n\}_{n=1}^\infty$. For each $t \in I$, let $g_0(t)$ be a best simultaneous approximation of $(f_1(t), f_2(t), ..., f_m(t))$ in $G$. Define $g_0$ from $I$ into $G$ such that $g_0(t)$ is a best simultaneous approximation of $(f_1(t), f_2(t), ..., f_m(t))$. Applying Lemma 3.3 to the mapping $g_0$ in each $I_n$ taking $\epsilon = \frac{1}{2}$ and $I = A = A_n$. We get a countable partition in each $I_n$ and therefore a countable partition in the whole of $I$. Thus we get a countable partition $\{E_n\}_{n=1}^\infty$ of $I$ in measurable sets and a sequence of subsets $\{A_n\}_{n=1}^\infty$ of $I$ such that

$$A_n \subseteq E_n, \mu^*(A_n) = \mu(E_n) < +\infty,$$

$$\text{diam}_\sigma(g_0(A_n)) < \frac{1}{2}, \text{diam}_\sigma(f_i(E_n)) < \frac{1}{2}, i = 1, 2, ..., m.$$  

Let us apply again the same argument in each $E_n$ with $\epsilon = \frac{1}{2^2}$, $I = E_n$ and $A = A_n$. For each $n$ we get a countable partition $\{E_{nk} : 1 \leq k < \infty\}$ of $E_n$ in measurable sets and a sequence $\{A_{nk} : 1 \leq k < \infty\}$ of subsets of $I$ such that

$$A_{nk} \subseteq E_{nk} \cap A_n, \mu^*(A_{nk}) = \mu(E_{nk}),$$

$$\text{diam}_\sigma(g_0(A_{nk})) < \frac{1}{2^2} \text{ and } \text{diam}_\sigma(f_i(E_{nk})) < \frac{1}{2^2}, i = 1, 2, ..., m,$$

for all $n$ and $k$. Let us proceed by induction. Now for each natural number $k$, let $\Delta_k$ be the set of $k$-tuples of natural numbers and let $\Delta = \bigcup_{k=1}^\infty \Delta_k$. On this $\Delta$ consider the partial order defined by $(m_1, m_2, ..., m_i) \leq (n_1, n_2, ..., n_j)$ if and only if $i \leq j$ and $m_k = n_k$.
for \( k = 1, 2, \ldots, i \). Then by induction for each natural number \( k \), we can take a partition \( \{E_\alpha : \alpha \in \Delta_k\} \) of subsets of \( I \) and a collection \( \{A_\alpha\}_{\alpha \in \Delta_k} \) such that:

1. \( A_\alpha \subseteq E_\alpha \) and \( \mu^*(A_\alpha) = \mu(E_\alpha) \) for each \( \alpha \).
2. \( A_\alpha \subseteq A_\beta \) and \( E_\alpha \subseteq E_\beta \) if \( \beta \leq \alpha \).
3. \( \text{diam}_\varphi(f_i(E_\alpha)) < \frac{1}{2^n} \) for \( i = 1, 2, \ldots, m \) and \( \text{diam}_\varphi(g_0(A_\alpha)) < \frac{1}{2^n} \) if \( \alpha \in \Delta_k \).

We may assume that \( A_\alpha \neq \emptyset \) for all \( \alpha \) (forget the \( \alpha \)'s for which \( A_\alpha = \emptyset \)). For each \( \alpha \in \Delta \) take \( t_\alpha \in A_\alpha \) and define \( g_k \) from \( I \) into \( G \) by \( g_k(.) = \sum_{\alpha \in \Delta_k} \varphi_{E_\alpha}(.)g_0(t_\alpha) \). Then for each \( t \in I \) and \( n \leq k \) we have:

\[
\varphi\|g_n(t) - g_k(t)\| = \varphi \left\| \sum_{\alpha \in \Delta_n} \varphi_{E_\alpha}(t)g_0(t_\alpha) - \sum_{\beta \in \Delta_k} \varphi_{E_\beta}(t)g_0(t_\beta) \right\|.
\]

But since \( n \leq k \) by 1 and 2 we have:

\[
\varphi\|g_n(t) - g_k(t)\| \leq \varphi \left\| \sum_{\beta \in \Delta_k} \varphi_{E_\beta}(t)\left( g_0(t_\alpha) - g_0(t_\beta) \right) \right\|
\leq \sum_{\beta \in \Delta_k} \phi \|g_0(t_\alpha) - g_0(t_\beta)\| \mu(E_\beta)
\leq \frac{1}{2^n}.
\]

Therefore \( (g_k(t)) \) is a Cauchy sequence in \( X \) for every \( t \in I \). Consequently \( (g_k(t)) \) is a convergent sequence for every \( t \in I \). Let \( g : I \to G \) be the point wise limit of \( (g_k) \). Since \( g_k \) is measurable for each \( k \), \( g \) is measurable. Let \( t \in I \) and let \( n \) be a natural number. Suppose \( t \in E_\alpha \). We have:

\[
\sum_{i=1}^{m} \varphi\|f_i(t) - g_n(t)\| = \sum_{i=1}^{m} \varphi\|f_i(t) - g_0(t_\alpha)\|
\leq \sum_{i=1}^{m} \varphi\|f_i(t) - f_i(t_\alpha)\| + \varphi\|f_i(t_\alpha) - g_0(t_\alpha)\|
\leq \sum_{i=1}^{m} \frac{1}{2^n} + \varphi\|f_i(t_\alpha) - g_0(t_\alpha)\|
\leq \frac{m}{2^n} + \text{dist}_\varphi((f_1(t_\alpha), f_2(t_\alpha), \ldots, f_m(t_\alpha)), G)
\leq \frac{m}{2^n} + \sum_{i=1}^{m} \varphi\|f_i(t) - f_i(t_\alpha)\|
+ \text{dist}_\varphi((f_1(t), f_2(t), \ldots, f_m(t)), G)
\leq \frac{m}{2^n-1} + \text{dist}_\varphi((f_1(t), f_2(t), \ldots, f_m(t)), G).
\]
Letting $n \to \infty$ we get:

$$\sum_{i=1}^{m} \varphi \| f_i(t) - g(t) \| = \lim_{n \to \infty} \sum_{i=1}^{m} \varphi \| f_i(t) - g_n(t) \|$$

$$= \text{dist}_{\varphi}(\{f_1(t), f_2(t), \ldots, f_m(t)\}, G).$$

and so $g(t)$ is a best simultaneous approximation of $f_1(t), f_2(t), \ldots, f_m(t)$ in $G$. □

**Theorem 3.5.** Let $\varphi$ be a modulus function and $G$ be a closed separable subspace of $X$. Then $L^\varphi(I, G)$ is simultaneously proximinal in $L^\varphi(I, X)$ if and only if $G$ is simultaneously proximinal in $X$.

**Proof.** Necessity is in Theorem 3.2 Let us show sufficiency. Suppose that $G$ is simultaneously proximinal in $X$, and let $f_1, f_2, \ldots, f_m$ be functions in $L^\varphi(I, X)$. Theorem 3.4 guarantees that there exists a measurable function $g$ defined on $I$ with values in $X$ such that $g(t)$ is a best simultaneous approximation of $f_1(t), f_2(t), \ldots, f_m(t)$ in $G$ for almost all $t$. It follows from Corollary 2.2 that $g$ is a best simultaneous approximation of $f_1, f_2, \ldots, f_m$ in $L^\varphi(I, G)$ □

**Theorem 3.6.** Let $\varphi$ be a modulus function. Then if $g \in L^\varphi(I, G)$ is a best simultaneous approximation from $L^\varphi(I, G)$ of an $m$-tuple of elements $f_1, f_2, \ldots, f_m \in L^\varphi(I, X)$ then for every measurable subset $A$ of $I$ and every $h \in L^\varphi(I, G)$,

$$\int_A \varphi (\| f_{j_0}(s) - g(s) \|) \, d\mu(s) \leq \int_A \varphi (\| f_{j_0}(s) - h(s) \|) \, d\mu(s),$$

for some $j_0 \in \{1, 2, \ldots, m\}$.

**Proof.** If $\mu(A) = 0$ then there is nothing to prove. Suppose that for some $A$ satisfying $\mu(A) > 0$ and for some $h_0 \in L^\varphi(I, G)$, the inequality does not hold for $J = 1, 2, \ldots, m$. Now, define $g_0 \in L^\varphi(I, G)$ by

$$g_0(s) := \begin{cases} g(s) & \text{if } s \in I - A \\ h_0(s) & \text{if } s \in A \end{cases}$$
Then we have for $j = 1, 2, \ldots, m$

\[
\int_I \varphi (\|f_j(s) - g_0(s)\|) \, d\mu = \int_A \varphi (\|f_j(s) - h_0(s)\|) \, d\mu(s)
+ \int_{I-A} \varphi (\|f_j(s) - g(s)\|) \, d\mu(s)
< \int_A \varphi (\|f_j(s) - g(s)\|) \, d\mu(s)
+ \int_{I-A} \varphi (\|f_j(s) - g(s)\|) \, d\mu(s)
= \int_I \varphi (\|f_j(s) - g(s)\|) \, d\mu(s).
\]

This implies that

\[
\sum_{j=1}^m \|f_j - g_0\|_\varphi < \sum_{j=1}^m \|f_j - g\|_\varphi
\]

which contradict the fact that $g$ is a best simultaneous approximation from $L^\varphi(I, G)$ of the m-tuple of elements $f_1, f_2, \ldots, f_m$. □

As a corollary we get:

**Corollary 3.7.** If $g$ is a best simultaneous approximation from $L^\varphi(I, G)$ of an m-tuple of elements $f_1, f_2, \ldots, f_m \in L^\varphi(I, X)$ then, for every measurable subset $A$ if $I$,

\[
\int_A \varphi (\|g(s)\|) \, d\mu(s) \leq 2 \max_{1 \leq j \leq m} \left( \int_A \varphi (\|f_j(s)\|) \, d\mu(s) \right).
\]

**Proof.** Since, for $j = 1, 2, \ldots, m$

\[
\int_A \varphi (\|g(s)\|) \, d\mu(s) \leq \int_A \varphi (\|f_j(s) - g(s)\|) \, d\mu(s) + \int_A \varphi (\|f_j(s)\|) \, d\mu(s),
\]

we obtain, by using Theorem 3.6 with $h = 0$, that for $j_0 \in \{1, 2, \ldots, m\}$

\[
\int_A \varphi (\|g(s)\|) \, d\mu(s) \leq 2 \int_A \varphi (\|f_{j_0}(s)\|) \, d\mu(s)
\leq 2 \max_{1 \leq j \leq m} \left( \int_A \varphi (\|f_j(s)\|) \, d\mu(s) \right),
\]

which completes the proof. □
We end this paper with the following result on best simultaneous approximation of $l^\varphi(X)$ in $l^\varphi(G)$.

**Theorem 3.8.** Let $\varphi$ be a modulus function. Then $l^\varphi(G)$ is simultaneously proximinal in $l^\varphi(X)$ if $G$ is simultaneously proximinal in $X$.

**Proof.** Let $f_1, f_2, \ldots, f_m \in l^\varphi(X)$. Since $G$ is simultaneously proximinal in $X$, for each $n$, there exists $g(n) \in G$ such that for every $y \in G$

$$\sum_{j=1}^{m} \varphi \| f_j(n) - g(n) \| \leq \sum_{j=1}^{m} \varphi \| f_j(n) - y \| .$$

Since $y = 0 \in G$, we get

$$\sum_{j=1}^{m} \varphi \| f_j(n) - g(n) \| \leq \sum_{j=1}^{m} \varphi \| f_j(n) \| .$$

But $\varphi$ is increasing and subadditive so

$$m \varphi \| g(n) \| = \sum_{j=1}^{m} \varphi \| g(n) - f_j(n) + f_j(n) \|
\leq \sum_{j=1}^{m} \varphi \| g(n) - f_j(n) \| + \varphi \| f_j(n) \| \leq 2 \sum_{j=1}^{m} \varphi \| f_j(n) \| .$$

Consequently $g = (g(n)) \in l^\varphi(G)$. We claim that $g$ is a best simultaneous approximation for $f_1, f_2, \ldots, f_m \in l^\varphi(X)$ in $l^\varphi(G)$. To see that let $h \in l^\varphi(G)$. Then

$$\sum_{j=1}^{m} \| f_j - h \|_\varphi = \sum_{j=1}^{m} \sum_{n=1}^{\infty} \varphi \| f_j(n) - h(n) \|
= \sum_{n=1}^{\infty} \sum_{j=1}^{m} \varphi \| f_j(n) - h(n) \|
\geq \sum_{n=1}^{\infty} \sum_{j=1}^{m} \varphi \| f_j(n) - g(n) \|
= \sum_{j=1}^{m} \sum_{n=1}^{\infty} \varphi \| f_j(n) - g(n) \|
= \sum_{j=1}^{m} \| f_j - g \|_\varphi .$$

### References


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